Level statistics for nearly integrable systems

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We assume that the level spectra of quantum systems in the initial phase of transition from integrability to chaos are approximated by superpositions of independent sequences. Each individual sequence is modeled by a random matrix ensemble. We obtain analytical expressions for the level spacing distribution and level number variance for such a system. These expressions are successfully applied to the analysis of the resonance spectrum in a nearly integrable microwave billiard.

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Random matrix theory (RMT) provides a framework for describing the statistical properties of spectra for quantum systems whose classical counterpart is chaotic [1,2]. It models the Hamiltonian of the system by an ensemble of random matrices, subject to some general symmetry constraints. Time-reversal-invariant quantum system is represented by a Gaussian orthogonal ensemble (GOE) of random matrices when the system has rotational symmetry and by a Gaussian symplectic ensemble otherwise. Chaotic systems without time-reversal symmetry are represented by the Gaussian unitary ensemble (GUE). A complete discussion of the level correlations even for these three canonical ensembles is a difficult task. Most of the interesting results are obtained for the limit of large matrices. Analytical results have long ago been obtained for the case of two-dimensional matrices [3]. It yields simple analytical expressions for the nearest-neighbor-spacing distribution (NNSD) renormalized to make the mean spacing equal one. The spacing distribution for the GOE, $p(s)=\frac{\pi}{2}s\exp(-\frac{\pi}{4}s^2)$, where s is the spacing between adjacent energy levels rescaled to unit mean spacing D, is known as Wigner's surmise. Analogous expression $p(s)=\frac{32}{\pi^2}s^2\exp(-\frac{4}{\pi}s^2)$ is obtained for the GUE [3,4].

There are elaborate theoretical arguments by Berry and Tabor [5] that classically integrable systems should have Poissonian statistics. The Poisson distribution of the regular spectra has been proved in some cases (see results by Sinai [6] and Marklof [7], for instance). Still its mechanism is not completely understood. It has also been confirmed by many numerical studies although the deviations of the calculated $P(s)$ from $\exp(-s)$ are often statistically significant (see [8] and references therein). The appearance of the Poisson distribution is now admitted as a universal phenomenon in generic integrable quantum systems.

A typical Hamiltonian system shows a phase space in which regions of regular motion and chaotic dynamics coexist. These systems are known as mixed systems. Their dynamical behavior is by no means universal, as is the case for fully regular and fully chaotic systems [9–11]. The nature of the mixed systems is more obscure in quantum than in classical mechanics. The assumptions that lead to the RMT description do not apply to mixed systems. While some elements of the Hamiltonian of a typical mixed system could be described as randomly distributed, the others would be nonrandom. Moreover, the matrix elements need not all have the same distributions and may or may not be correlated. Thus, the RMT approach is a difficult route to follow. Comprehensive semiclassical computations have been carried out for Hamiltonian quantum systems, which on the classical level have a mixed phase space dynamics (see, e.g., [12] and references therein). Berry and Robnik [13] elaborated a NNSD for mixed systems based on the assumption that semiclassically the eigenfunctions and associated Wigner distributions are localized either in classically regular or chaotic regions in phase space. Accordingly, the sequences of eigenvalues connected with these regions are assumed to be statistically independent, and their mean spacing is determined by the invariant measure of the corresponding regions in phase space. There have been several proposals for phenomenological random matrix theories that interpolate between the Wigner-Dyson RMT and banded random matrices with an almost Poissonian spectral statistics [14]. Unfortunately, these works do not lead to valid analytical results, which make them difficult to use in the analysis of experimental data. There are other phenomenological approaches (see, e.g., [15] and references therein), which use nonextensive statistical mechanics, based on maximizing the Tsallis or Kaniadakis entropies, as well as the recently proposed concept of superstatistics. These approaches have the advantage of conserving base invariance of the Hamiltonian matrix. They provide a satisfactory description near the end of transition from integrability to chaos.

This Brief Report considers another phenomenological approach, which has the spirit of the Kolmogorov-Arnold-Moser (KAM) theorem, to the stochastic transition in quantum systems. The phase space of the integrable system consists of infinitely many tori corresponding to the conserved symmetries of the system. In the semiclassical limit, energy eigenstates are expected to be localized on individual tori. Tori destruction corresponds to the mixing of the corresponding quantum eigenstates. Symmetry breaking breaks some of the invariant tori but only deforms others according to the KAM theorem. Quantum symmetry breaking strongly mixes a limited number of eigenstates but has a less influence on the other ones. Thus, the spectrum is divided into independent sequences of eigenvalues. States belonging to the same sequence are strongly mixed. The sequence may be modeled by a GOE if time-reversal invariance is preserved. The interaction between states belonging to different sequences grows as symmetry breaking increases. This amounts to amalgamating the initial sequences into a fewer number of independent sequences with no more regular character. Conse-

quently, as the number of the no-symmetry sequences decreases, their fractional density increases accordingly. As the state of chaos is reached, the whole spectrum consists of a single (GOE) sequence.

Abul-Magd and Simbel [16] consider another class of mixed systems, in which the degrees of freedom are divided into two noninteracting groups, one having chaotic dynamics and one regular. The Hamiltonian of such a system is given as a sum of two terms so that each of the eigenvalues of the total Hamiltonian is expressed as a superposition of two eigenvalues corresponding to the two Hamiltonian terms. The spectrum is then given by a superposition of independent chaotic subspectra. Each subspectrum corresponds to one (or one set) of the quantum numbers of the regular component of the Hamiltonian. This model is used in [17] to describe level statistics of vibrational nuclei. An elaborated version of this model [18] has been applied to study NNSD of a wide range of nuclei [19].

We shall now consider the energy spectra of nearly integrable systems that may be represented as a superposition of independent sequences S_i each having fractional level density f_i, with $i=1,\ldots,m$, and with $\Sigma_{i=1}^{m} f_i=1$. In this case, NNSD of the composite spectrum can be exactly expressed in terms NNSD's of the constituting sequences (see, e.g., Appendix A.2 of Mehta's book [1]). The gap probability function

$$E(s) = \int_s^\infty ds' \int_{s'}^\infty ds'' p(s') \tag{1}$$

that gives the probability of finding no eigenvalues in segment of length s of the total spectrum, is expressed as a product of the gap functions of the individual sequences

$$E(m,s) = \prod_{i=1}^{m} E_i(f_i s). \tag{2}$$

We assume that all of S_i obey the GOE statistics. Then NNSD of each of the individual sequences distribution is given by the Wigner surmise, then for all i

$$E_i(x) = E_{\mathrm{GOE}}(x) = \mathrm{Erfc}\left(\frac{\sqrt{\pi}}{2}x\right), \tag{3}$$

where $\mathrm{Erfc}(x)$ is the complimentary error function. The NNSD of the full spectrum can the be obtained by twice differentiating the resulting gap function.

We shall restrict our consideration to the case when all sequences have the same fractional level density so that $f_i = f = 1/m$. The gap function of the composite spectrum is then given by

$$E(m,s) = \left[\mathrm{Erfc}\left(\frac{\sqrt{\pi}}{2m}s\right)\right]^m. \tag{4}$$

Differentiating this function twice with respect to s, we obtain

$$p(m,s) = \left[\mathrm{Erfc}\left(\frac{\sqrt{\pi}}{2m}s\right)\right]^{m-2} e^{-\pi s^2/4m^2} \times \left[\left(1-\frac{1}{m}\right)e^{-\pi s^2/4m^2} + \frac{\pi s}{2m^2}\mathrm{Erfc}\left(\frac{\sqrt{\pi}}{2m}s\right)\right]. \tag{5}$$

It is easy to see that for a single sequence $p(1,s)=\frac{\pi}{2}s \exp(-\frac{\pi}{4}s^2)$ so that the Wigner surmise is recovered. On the other hand, $\lim_{m\to\infty} p(m,s)=e^{-s}$ as required.

A weak point of the distribution in Eq. (5) is that it differs from zero at $s=0$ because the symmetry-breaking interaction lifts the degeneracies. The model thus fails in the domain of small spacings as far as the NNSDs are concerned. The magnitude of this domain depends on the ratio of the strength of the symmetry-breaking interaction to the mean level spacing. Therefore, it is expected to work well for nearly integrable system.

In the case when the individual sequences are described by a GUE, the gap function for each individual sequence is given by

$$E_{\mathrm{GUE}}(x) = e^{-4x^2/\pi} - x\,\mathrm{Erfc}\left(\frac{2}{\sqrt{\pi}}x\right). \tag{6}$$

In this case, the NNSD of the composite spectrum is given by

$$p(m,s) = \frac{1}{m^3}\left[e^{-4s^2/\pi m^2} - \frac{s}{m}\mathrm{Erfc}\left(\frac{2s}{m\sqrt{\pi}}\right)\right]^{m-2} \times \left\{(m-1)\left[\frac{4s}{\pi}e^{-4s^2/\pi m^2} + m\,\mathrm{Erfc}\left(\frac{2s}{m\sqrt{\pi}}\right)\right]^2 + \frac{32s^2}{\pi^2}e^{-4s^2/\pi m^2}\left[e^{-4s^2/\pi m^2} - \frac{s}{m}\mathrm{Erfc}\left(\frac{2s}{m\sqrt{\pi}}\right)\right]\right\}. \tag{7}$$

The situation with the level number variance (LNV) Σ^2 of composite spectra is not as clear as in the case of NNSD. Seligman and Verbaarschot [20] argued that Σ^2 is a variance and can therefore be expressed for a composite spectrum as a sum of the corresponding quantities for its subspectra,

$$\Sigma^2(m,L) = \sum_{i=1}^{m} \Sigma_i^2(f_i L), \tag{8}$$

where $\Sigma_i^2(x)$ is the LNV of the ith sequence. There, the LNV of the composite spectrum composed of m independent sequences described by RMT is given by

$$\Sigma^2(m,L) = m\Sigma_{\mathrm{RMT}}^2(L/m), \tag{9}$$

where $\Sigma_{\mathrm{RMT}}^2(L)$ is the LNV calculated by RMT. Explicit expressions for $\Sigma_{\mathrm{RMT}}^2(L)$ in the cases of GOE and GUE are given in Mehta's book.

We shall compare our predictions for the NNSD and the LNV with the energy spectra of a Limaçon billiard. This is a closed billiard whose boundary is defined by the quadratic conformal map of the unit circle z to w,

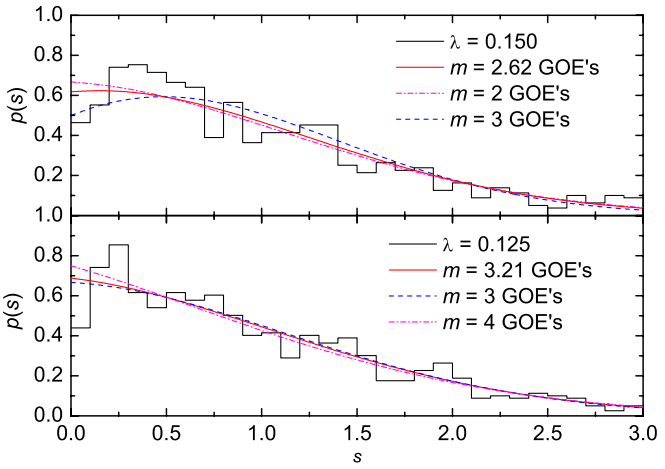

FIG. 1. (Color online) Comparison between the NNSD for two nearly integrable Limaçon billiards with the same distribution for m-independent GOE sequences.

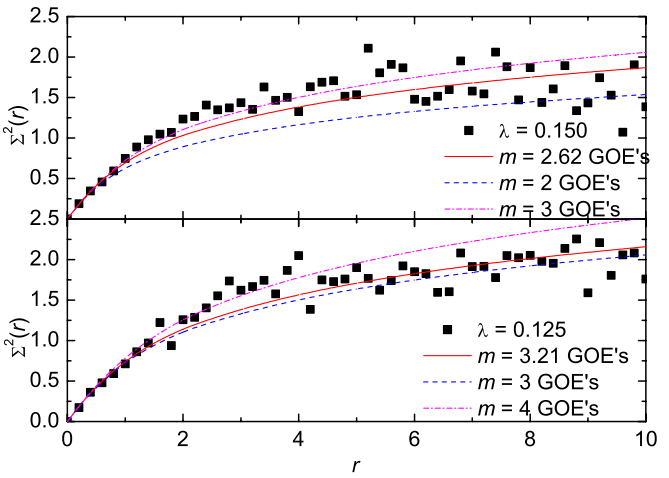

FIG. 2. (Color online) Comparison between the LNV for two nearly integrable Limaçon billiard with the same distribution for m-independent GOE sequences.

$$w = z + \lambda z^2, \quad |z| = 1. \tag{10}$$

The shape of the billiard is controlled by a single parameter λ. For $0 \leq \lambda < 1/4$, the Limaçon billiard has a continuous and convex boundary with a strictly positive curvature and a collection of caustics near the boundary. At $\lambda = 1/4$, the boundary has zero curvature at its point of intersection with the negative real axis, which turns into a discontinuity for $\lambda > 1/4$. The classical dynamics of this system and the corresponding quantum billiard have been extensively investigated by Robnik *et al.* [21]. They concluded that the dynamics in the Limaçon billiard undergoes a smooth transition from integrable motion at $\lambda = 0$ via a soft chaos KAM regime for $0 < \lambda \leq 1/4$ to a strongly chaotic dynamics for $\lambda = 1/2$. We assume that the quantum dynamics of the Limaçon billiard can approximately be described by the model present here. The spherical billiard for which $\lambda = 0$ has two good quantum numbers, namely, the energy and angular momentum. As λ increases, the spherical symmetry is gradually destroyed. States corresponding to different angular-momentum quantum numbers mix to different degrees depending on the magnitude of the wave functions at large z.

The resonance spectra in microwave cavities with the shape of billiards from the family of Limaçon billiards have been constructed for the values $\lambda = 0.125, 0.150, 0.300$ and the first 1163, 1173, and 942 eigenvalues were measured, respectively [22,23]. The billiard with $\lambda = 0.300$ has a chaotic dynamics, and its resonance spectrum is well described by a GOE [24], i.e., using Eqs. (5) and (9) with $m = 1$. We here consider the $\lambda = 0.125, 0.150$ billiards that exhibit mixed regular-chaotic dynamics, which is predominantly regular. We have performed a least-squares analysis of the NNSD and LNV for these billiards using Eqs. (5) and (9), respectively, taking m as a real parameter. The best fit values are $m = 3.21, 2.62$ for the billiards with $\lambda = 0.125, 0.150$, respectively. Figure 1 shows the result of comparison of the experimental NNSD for these billiards with Eq. (5), while the result for the LNV is given in Fig. 2. We note, however, the interpretation of m as the number of spectra that are being superimposed suggests that it should be an integer. For this reason, we show in the two figures the results of calculation with 3 and 4 sequences for the $\lambda = 0.125$ billiard and 2 and 3 sequences for the $\lambda = 0.150$ one. The figures show that the agreement with the fractional value of m is not so much better than with the integer values of the parameter. In both cases, the parameter m can be taken equal to 3 for both the NNSD and LNV in spite of the fact that the NNSD is close to a Poisson distribution while the LNV shows a large amount of spectral rigidity. This unusual situation is in favor of the validity of the present model. To demonstrate this we show an analysis of the NNSD in Fig. 3 and of the LNV in Fig. 4 using the Berry-Robnik model [13]. The best-fit value of the parameter q that measures the fractional volume of the regular part of the phase space is found for the NNSD to be 0.585. This is quite different from the value 0.156 that fits the LNV. We note that the agreement between the prediction of the Berry-Robnik model and the experimental LNV is worse than the agreement with our model. Concerning the NNSD, both models fail to describe the depletion in the

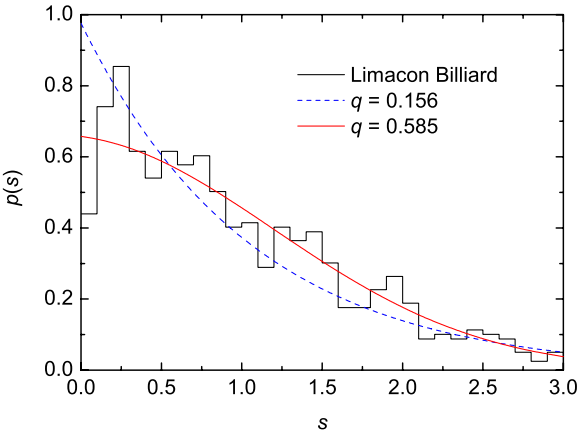

FIG. 3. (Color online) Comparison between the NNSD for a nearly integrable Limaçon billiard ($\lambda = 0.125$) with the same distribution calculated using the Berry-Robnik semiclassical method.

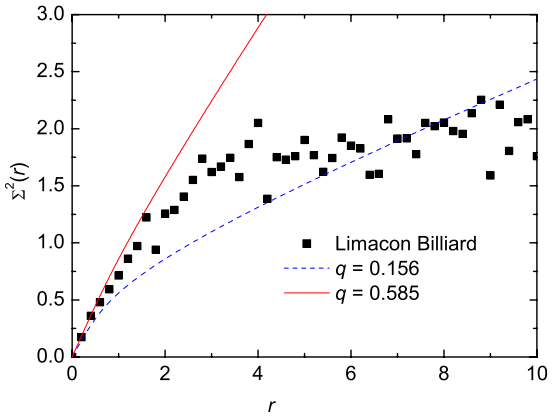

FIG. 4. (Color online) Comparison between the LNV for a nearly integrable Limaçon billiard ($\lambda=0.125$) with the same distribution calculated using the Berry-Robnik semiclassical method.

number of events in the first bin. There is 100 spacings in this bin so that the statistical error is 10%. Thus the depletion is statistically significant. The disagreement reflects the partial neglect of level repulsion in both model where the superimposed sequences are considered as independent.

The expression for NNSD of a spectrum composed of independent sequences with nonequal fractional densities f_i is more complicated. It has been shown in [16,18] that the NNSD in this case essentially depends on a single parameter, namely, $\langle f\rangle=\Sigma_{i=1}^{m}f_i^2$, which is the mean fractional level density for the superimposed sequences; the statistical weight of each sequence is given again by its fractional density. For a superposition of equal sequences, $f_i=\langle f\rangle=1/m$. Therefore, Eq. (5) can approximately be used to describe a superposition of independent but not equal sequences by considering m as a parameter, not necessarily taking integer values. The noninteger parameter m will play the role of an effective number of the constituting sequences $m=1/\langle f\rangle$. One can adopt this interpretation of the parameter m if one sees that the fit in Figs. 1 and 2 are deteriorated by taking $m=3$ instead of 3.21 or 2.62,

To summarize, we consider a model for systems with regular-chaotic dynamics in which the energy spectrum is represented by an independent sequences of levels, each one modeled by a Gaussian random ensemble. By varying the effective number of sequences, the model interpolates between the Poissonian spectrum for the regular system where the spectrum consists of infinite number of sequences and that of a chaotic system whose spectrum consists of a single sequence. We show that the model successfully describe both the NNSD and LNV for a nearly integrable Limaçon billiard with the same value of the model parameter.

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